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On the null-field equations for water-wave radiation problems

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Consider a rigid body which is performing simple harmonic oscillations of small amplitude in the free surface of deep water under gravity. Under certain geometrical conditions on ∂D, the wetted surface of the body, it is known that the linear boundary-value problem $\mathscr{P}$ for a corresponding velocity potential ϕ is uniquely solvable at all frequencies. The usual method for solving $\mathscr{P}$ is to derive a Fredholm integral equation of the second kind over ∂D. There are two familiar ways of doing this: (i) represent ϕ as a distribution of simple wave sources over ∂D, leading to an integral equation for the unknown source strength; (ii) apply Green's theorem to ϕ and a simple wave source; when the field point lies on ∂D, this gives an integral equation for the boundary values of ϕ. It is well known that both of these integral equations have unique solutions, except at the same infinite discrete set of frequencies (the irregular frequencies).

In this paper, we shall describe an alternative method for solving $\mathscr{P}$: when the field point, in (ii), lies inside the body, we obtain an integral relation. If the simple wave source has a suitable bilinear expansion, this integral relation may be reduced to an infinite set of equations for the boundary values of ϕ. These equations, called the 'null-field equations for water waves', appear to be new; equations of this type were first obtained by Waterman for electromagnetic and acoustic scattering problems. The required bilinear expansion has been given by Ursell (1981) for two dimensions, and is given here for three dimensions. Using these, we show that the null-field equations always have a unique solution – irregular frequencies do not occur. This result is proved here for water waves in two and three dimensions. Similar results may be obtained for water of constant finite depth.

1. Introduction

Consider a rigid body which is floating in the free surface of a fluid. We suppose that the fluid is incompressible, inviscid and of infinite depth. We denote the fluid domain by D, the free surface by F and the wetted surface of the body by ∂D, which we assume has the following properties (John 1950). Let ∂D^* denote the union of the surface ∂D and its mirror image in the plane of the free surface. We shall say that ∂D has properties J if ∂D^* is a convex, twice-differentiable surface. (This implies that ∂D must intersect the free surface perpendicularly.)

We take Cartesian co-ordinates with the y-axis vertical (y increasing with depth) such that F occupies a region of the plane $y = 0$.

Suppose that the body performs simple harmonic oscillations of small amplitude and radian frequency ω. For irrotational motion, we can formulate the following well-known, linear boundary-value problem $\mathscr{P}$ for a velocity potential $\mathscr{R}\{\phi(P)\, e^{-i\omega t}\}$:

Determine a function $\phi(P)$ satisfying Laplace's equation

$$\nabla^2\phi(P) = 0 \quad \text{in } D,$$

the free-surface condition

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \quad \text{on } F \tag{1.1}$$

and the boundary condition

$$\frac{\partial \phi(p)}{\partial n_p} = V(p) \quad \text{on } \partial D, \tag{1.2}$$

where $K = \omega^2/g$, g is the acceleration due to gravity, and the function $V(p)$ is prescribed on ∂D. In addition, there is the radiation condition that waves travel outwards to infinity, and the condition that the fluid motion vanishes as $y \to \infty$.

The notation is as follows: capital letters P, Q denote points of D; lower-case letters p, q denote points of ∂D; the origin O is assumed to lie in F_-, the portion of the plane $y = 0$ which is inside the body; D_- denotes the interior of the body, i.e. the region with boundary $\partial D \cup F_-$; P_-, Q_- denote points of D_-; r_P is the length OP; $\partial/\partial n_p$ denotes normal differentiation at the point p, in the direction from ∂D into D.

The following theorem on the solvability of $\mathscr{P}$ has been proved by John (1950).

Theorem 1. Suppose that ∂D has properties J and that $V(q)$ is continuous on ∂D. Then there exists a unique solution of the boundary-value problem $\mathscr{P}$, for all real values of K.

We shall henceforth assume that the conditions of theorem 1 are always satisfied.

The usual approach for solving the boundary-value problem $\mathscr{P}$ is to derive an integral equation of the second kind, over the boundary ∂D. One way of doing this is to assume that $\phi(P)$ can be represented by a distribution of sources over ∂D; the source strength is then found to be the solution of a Fredholm integral equation of the second kind. Alternatively, an integral equation of the second kind can be derived directly from Green's theorem. It is well known that both of these methods (which will be described in § 2) lead to boundary integral equations of the second kind which are singular at a certain infinite discrete set of frequencies, corresponding to the eigenvalues of a related interior problem. This phenomenon is clearly a consequence of the method of solution, for we have already remarked that the original boundary-value problem $\mathscr{P}$ is known to have a unique solution at all frequencies, provided that ∂D has properties J and $V(q)$ is continuous (theorem 1).

A different approach to the related problems in acoustics has been employed by Waterman (1969). His method is based on solving the Helmholtz formula in the interior of the body and leads to an infinite system of equations, rather than a single (integral) equation. Martin (1980) has studied these equations (called the 'null-field equations of acoustics'), and proved that they always have a unique solution, i.e. difficulties at interior eigenvalues do not occur with this method.

In this paper, we shall prove some analogous results for water-wave problems. In § 3, we derive the null-field equations for water waves in two dimensions, and prove that they always have a unique solution. In § 4, we consider the special case of an oscillating half-immersed circular cylinder. For this geometry, we show that the null-field equations may be obtained by suitably modifying Ursell's well-known method of multipoles. In § 5, we suggest a method which can be used to solve the null-field equations numerically. As an example, we use this method to solve the equations corresponding to a heaving, half-immersed, elliptic cylinder. Finally, in § 6, we derive the null-field equations for water waves in three dimensions. Again, we are able to establish existence and uniqueness at all frequencies. The proof requires the extension of some results of Ursell (1981) to three dimensions, and these are given in appendix B; in particular, we give a new expansion for the simple wave source in three dimensions.

2. Boundary integral equations

Let $G(P,Q)$ be any fundamental solution, i.e. $G(P,Q)$ satisfies Laplace's equation in D, the free-surface condition (1.1) and the radiation condition at infinity, and has a suitably normalized source singularity at Q (see e.g. Ursell 1981). The simplest choice for $G(P,Q)$ is the simple wave source (Thorne 1953); in two dimensions (where the motion is independent of z), we have

$$G_0(P,Q) \equiv G_0(x,y;\xi,\eta) = \tfrac{1}{2}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2} - 2\int_0^\infty e^{-k(y+\eta)}\cos k(x-\xi)\frac{dk}{k-K}, \tag{2.1a}$$

whilst in three dimensions we have

$$\begin{aligned} G_0(P,Q) &\equiv G_0(x,y,z;\xi,\eta,\zeta) \\ &= \tfrac{1}{2}\{R^2+(y-\eta)^2\}^{-\frac{1}{2}} + \tfrac{1}{2}\{R^2+(y+\eta)^2\}^{-\frac{1}{2}} + K\int_0^\infty e^{-k(y+\eta)}J_0(kR)\frac{dk}{k-K}, \end{aligned} \tag{2.1b}$$

where $R^2 = (x-\xi)^2+(z-\zeta)^2$ and, in order to satisfy the radiation condition, the path of integration passes below the pole of the integrand at $k = K$. (Note that our definition of $G_0(x,y,z;\xi,\eta,\zeta)$ differs from the usual definition by an extra factor of $\frac{1}{2}$ in (2.1b); this enables us to write down the same equations in three dimensions as in two dimensions.)

If we apply Green's theorem in D, to $\phi(P)$ and $G_0(P,Q)$, we obtain the following equations:

$$2\pi\phi(P) = \int_{\partial D}\left\{G_0(P,q)\frac{\partial}{\partial n_q}\phi(q) - \phi(q)\frac{\partial}{\partial n_q}G_0(P,q)\right\}ds_q, \tag{2.2}$$

$$\pi\phi(p) = \int_{\partial D}\left\{G_0(p,q)\frac{\partial}{\partial n_q}\phi(q) - \phi(q)\frac{\partial}{\partial n_q}G_0(p,q)\right\}ds_q, \tag{2.3}$$

$$0 = \int_{\partial D}\left\{G_0(P_-,q)\frac{\partial}{\partial n_q}\phi(q) - \phi(q)\frac{\partial}{\partial n_q}G_0(P_-,q)\right\}ds_q. \tag{2.4}$$

These equations are analogous to Helmholtz's formulae in acoustics (see, for example, Martin 1980). Similar equations may be derived when $G_0(P,Q)$ is replaced by any fundamental solution $G(P,Q)$.

If we use the boundary condition (1.2) in (2.3), we obtain

$$\pi\phi(p) + \int_{\partial D}\phi(q)\frac{\partial}{\partial n_q}G_0(p,q)\,ds_q = \int_{\partial D}G_0(p,q)\,V(q)\,ds_q, \tag{2.5}$$

which is a Fredholm integral equation of the second kind for the unknown boundary values of ϕ. This integral equation possesses a unique solution unless the corresponding homogeneous integral equation,

$$\pi\phi(p) + \int_{\partial D}\phi(q)\frac{\partial}{\partial n_q}G_0(p,q)\,ds_q = 0, \tag{2.6}$$

has a non-trivial solution. It was shown by John (1950) that (2.6) does have non-trivial solutions whenever K is an eigenvalue of the 'interior wave–Dirichlet problem', where the Dirichlet condition $\phi = 0$ is satisfied on ∂D and the free-surface condition (1.1) is satisfied on F_-. At such values of K (called *irregular* by John) the integral equation (2.5) does not have a unique solution for general $V(p)$.

A different approach for solving $\mathscr{P}$ is to represent $\phi(P)$ by a distribution of simple wave sources over ∂D,

$$\phi(P) = \int_{\partial D} \mu(q)\, G_0(P, q)\, ds_q. \tag{2.7}$$

On applying the boundary condition (1.2), we find that the unknown source strength $\mu(q)$ satisfies

$$\pi\mu(p) + \int_{\partial D} \mu(q) \frac{\partial}{\partial n_p} G_0(p, q)\, ds_q = V(p). \tag{2.8}$$

This integral equation is of the same form as (2.5), except that the kernel of (2.8) is the transpose of the kernel appearing in (2.5). (Here we have used the symmetry of the fundamental solution (2.1).) Hence (2.8) has the same irregular values as (2.5).

When K is not an irregular value, we can construct the solution of $\mathscr{P}$ by substituting the unique solution of (2.8) into (2.7). This is because the representation (2.7) satisfies Laplace's equation in D, the radiation condition and the free-surface condition (for any continuous $\mu(q)$), whilst it automatically satisfies the boundary condition (1.2) on ∂D if $\mu(q)$ satisfies (2.8). The situation is not so straightforward with the integral equation (2.5). Nevertheless, if we substitute the unique solution of (2.5) into (2.2), we can define a function $U(P)$, say, by

$$2\pi U(P) = \int_{\partial D} \left\{ G_0(P, q)\, V(q) - \phi(q) \frac{\partial}{\partial n_q} G_0(P, q) \right\} ds_q, \tag{2.9}$$

which, by the following theorem, solves $\mathscr{P}$.

Theorem 2. If K is not an irregular value and $\phi(q)$ is the unique solution of (2.5), the function $U(P)$, defined by (2.9), solves the boundary-value problem $\mathscr{P}$.

Proof. (Here we use arguments which are similar to those used by Kleinman & Roach (1974) in the proof of their theorem 5.1.) Clearly, $U(P)$ satisfies Laplace's equation in D, the radiation condition and the free-surface condition. It only remains to show that $U(P)$ also satisfies the boundary condition on ∂D, namely

$$\frac{\partial}{\partial n_p} U(p) = V(p). \tag{2.10}$$

If we differentiate (2.9) and let P approach ∂D, we see that $U(P)$ will satisfy (2.10) if $\phi(q)$ also satisfies the following *compatibility condition*:

$$\frac{\partial}{\partial n_p} \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} G_0(p, q)\, ds_q = -\pi V(p) + \int_{\partial D} V(q) \frac{\partial}{\partial n_p} G_0(p, q)\, ds_q. \tag{2.11}$$

(Note that a sufficient condition for the existence of the left-hand side of (2.11) is that ϕ be differentiable on ∂D, which is assumed to have properties J; see Kleinman & Roach (1974).)

Let us define a function in D_- by

$$U_0(P_-) = \int_{\partial D} \left\{ G_0(P_-, q)\, V(q) - \phi(q) \frac{\partial}{\partial n_q} G_0(P_-, q) \right\} ds_q. \tag{2.12}$$

U_0 satisfies Laplace's equation in D_- and the free-surface condition on F_-. Moreover, if we let P_- approach ∂D, we find that

$$\begin{aligned} U_0(p) &= -\pi\phi(p) + \int_{\partial D} \left\{ G_0(p,q)\, V(q) - \phi(q) \frac{\partial}{\partial n_q} G_0(p,q) \right\} ds_q \\ &= 0, \quad \text{by} \quad (2.5). \end{aligned}$$

Since K is not an irregular value, it follows that U_0 vanishes identically in D_- and, in particular,

$$\frac{\partial}{\partial n_p^-} U_0(p) = 0,$$

where $\mathbf{n}^-$ is the unit normal vector on ∂D pointing *into* D_-. Differentiating (2.12), we find that

$$\begin{aligned} 0 = \frac{\partial}{\partial n_p^-} U_0(p) = -\pi V(q) &+ \int_{\partial D} V(q) \frac{\partial}{\partial n_p} G_0(p,q)\, ds_q \\ &- \frac{\partial}{\partial n_p} \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} G_0(p,q)\, ds_q. \end{aligned}$$

Thus the compatibility condition (2.11) is satisfied and so (2.9) solves $\mathscr{P}$.

When K is an irregular value, the integral equations (2.5) and (2.8) are not uniquely solvable for general $V(p)$. This difficulty was first overcome by John (1950). He was able to prove that $\mathscr{P}$ is uniquely solvable at the irregular values by giving a rather complicated argument involving the non-trivial solutions of the homogeneous form of equation (2.8). Another way of overcoming the difficulty at the irregular values is to use a different fundamental solution in place of $G_0(P,Q)$; see § 3.

Numerical solutions of the integral equations (2.5) and (2.8) have been obtained by many authors, for various ∂D and $V(q)$. It is known that the discretized versions of these integral equations become ill-conditioned within a narrow band of frequencies around each irregular frequency. Although several computational devices have been used to overcome this difficulty, it is not pertinent to describe them all here; for a recent review see Mei (1978).

Let us now examine (2.4). This is an integral relation which asserts that the potential induced in D_- by the sources on ∂D is exactly cancelled by the potential induced by the dipoles on ∂D. Waterman (1969) calls this the 'extended boundary condition', and (2.4) the 'extended integral equation'. According to Mei (1978, p. 402), (2.4) has not been used for water-wave calculations. In acoustics, however, Waterman (1969) has replaced the interior integral relation (2.4) by an infinite system of equations, called the null-field equations. Martin (1980) has shown that the exterior problems of acoustics can always be solved by solving the system of null-field equations, i.e. the unphysical irregular values do not occur with this method. In § 3, we shall prove the corresponding result for water waves in two dimensions.

3. The null-field equations for water waves in two dimensions

Recently, Ursell (1981) has shown that the simple wave source in two dimensions, (2.1a), may be written as

$$G_0(P,Q) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^\sigma(P)\, \Phi_m^\sigma(Q), \tag{3.1}$$

for $r_Q > r_P$, where the functions α_m^σ and Φ_m^σ are defined in appendix A; α_m^σ are regular and satisfy the free-surface condition (1.1) whilst Φ_m^σ (which are usually known as 'multipole' potentials) satisfy the free-surface and radiation conditions, and are singular at O.

Let C_- be the inscribed circle to ∂D^*, which is centred on O. Similarly, let C_+ be the escribed circle to ∂D^*. Let D_N be the semicircular region which is bounded by F_- and the lower half of C_-; thus, D_N is contained in D_-. When P_- lies inside D_N (where $r_P < r_q$), we may substitute (3.1) into (2.4) to give

$$\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\alpha_m^\sigma(P_-)\int_{\partial D}\left\{\phi(q)\frac{\partial}{\partial n_q}\Phi_m^\sigma(q)-V(q)\,\Phi_m^\sigma(q)\right\}ds_q=0, \tag{3.2}$$

where we have used the boundary condition (1.2). Since the functions $\alpha_m^\sigma(P_-)$ are regular solutions of Laplace's equation in D_N, it follows that each term in (3.2) must vanish and so we obtain the following set of equations:

$$\int_{\partial D}\left\{\phi(q)\frac{\partial}{\partial n_q}\Phi_m^\sigma(q)-V(q)\,\Phi_m^\sigma(q)\right\}ds_q=0\quad(\sigma=1,2;\;m=0,1,2,\ldots).$$

We call these the 'null-field equations for water waves' and refer to them henceforth as (N.F.). These equations, which appear to be new, form an infinite set of equations for the boundary values of ϕ. Once ϕ is known on ∂D, we may use the integral representation (2.2), together with (1.2), to evaluate ϕ everywhere in D. In particular, if P lies outside C_+, we can use (3.1) in (2.2) to obtain

$$\phi(P)=\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}c_m^\sigma\,\Phi_m^\sigma(P), \tag{3.3}$$

where the coefficients c_m^σ are given by

$$2\pi c_m^\sigma=\int_{\partial D}\left\{\alpha_m^\sigma(q)\,V(q)-\phi(q)\frac{\partial}{\partial n_q}\alpha_m^\sigma(q)\right\}ds_q\quad(\sigma=1,2;\quad m=0,1,2,\ldots). \tag{3.4}$$

Equation (3.3) implies that, exterior to C_+, $\phi(P)$ has an expansion in terms of a set of functions $\{\Phi_m^\sigma\}$, each of which satisfies Laplace's equation in D, the free-surface condition and the radiation condition. Expansions of this type were first used by Ursell (1949) to solve the problem of the heaving, half-immersed circular cylinder (see § 4). Later, he proved that the set $\{\Phi_m^\sigma\}$ is complete (Ursell 1950). Therefore, we can *assume* that (3.3) holds exterior to C_+ and then proceed to give an alternative derivation of the null-field equations. Apply Green's theorem to $\phi(P)$ and $\Phi_m^\sigma(P)$ in the region bounded by ∂D, F and S, where S is a large semicircle, of radius r, enclosing ∂D and centred on O. There is no contribution from integrating over F, since ϕ and Φ_m^σ both satisfy (1.1). We can show that the contribution from integrating over S vanishes as $r\to\infty$, by using (3.3), and then using asymptotic properties of Φ_m^σ to prove that

$$\lim_{r\to\infty}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left\{\Phi_m^\sigma(r,\theta)\frac{\partial}{\partial r}\Phi_n^\nu(r,\theta)-\Phi_n^\nu(r,\theta)\frac{\partial}{\partial r}\Phi_m^\sigma(r,\theta)\right\}r\,d\theta=0. \tag{3.5}$$

(Here we have assigned plane polar co-ordinates (r,θ), $-\frac{1}{2}\pi\leqslant\theta\leqslant\frac{1}{2}\pi$, to points on S.) If we now choose appropriate values for m and σ, we obtain the complete set of null-field equations (N.F.).

Thus, we see that the null-field equations do not depend, essentially, on the bilinear expansion of the wave source (2.1*a*), or on the interior integral relation (2.4), but on the expansion of potentials which satisfy the free-surface and radiation conditions, as (3.3).

Let us conclude this section by proving that the set of null-field equations (N.F.) possesses a unique solution $\phi(q)$ for all values of K. We shall use arguments similar to those used by Martin (1980) for the corresponding exterior problem of acoustics.

Suppose we multiply each of (N.F.) by $\alpha_m^\sigma(P_-)$, where $P_- \in D_N$, and then use (3.1) to yield

$$U_0(P_-) = 0, \quad P_- \in D_N, \tag{3.6}$$

where U_0 is defined by (2.12). Since U_0 is a solution of Laplace's equation in D_- which vanishes in D_N, we can assert that (3.6) holds for all $P_- \in D_-$. Letting P_- approach ∂D, we obtain (2.5), which is a Fredholm integral equation of the second kind for $\phi(q)$. As we have already remarked, (2.5) has a unique solution except at the irregular values of K. Conversely, if we are not at an irregular value, it follows that the unique solution of (2.5) also solves the null-field equations. For, if $\phi(q)$ solves (2.5), we can define a function $U_0(P_-)$, by (2.12), which, by the arguments used to prove theorem 2, vanishes identically in D_-, and so $\phi(q)$ satisfies (N.F.).

At the irregular values of K, this argument must be modified. Multiply the first $N+1$ equations of (N.F.) by $a_m^\sigma \Phi_m^\sigma(P_-)$, where the a_m^σ are constants, and add the resulting equations to (N.F.), to give

$$U_1(P_-) \equiv \int_{\partial D} \left\{ G_1(P_-, q)\, V(q) - \phi(q) \frac{\partial}{\partial n_q} G_1(P_-, q) \right\} ds_q = 0, \tag{3.7}$$

where $G_1(P, Q)$ is a new (symmetric) fundamental solution, defined by

$$G_1(P, Q) = G_0(P, Q) + \sum_{m=0}^{N} \sum_{\sigma=1}^{2} a_m^\sigma\, \Phi_m^\sigma(P)\, \Phi_m^\sigma(Q). \tag{3.8}$$

Proceeding as before, we let P_- approach ∂D to obtain

$$\pi\phi(p) + \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} G_1(p, q)\, ds_q = \int_{\partial D} G_1(p, q)\, V(q)\, ds_q, \tag{3.9}$$

which is another integral equation of the second kind for $\phi(q)$.

Fundamental solutions of the form (3.8) have been considered by Ursell (1953, 1981) and Sayer (1980). Ursell (1953) solved the problem of a half-immersed circular cylinder, of radius a, which is performing high-frequency, vertical oscillations on deep water; by taking $N = 0$ and a particular value for a_0^1, he obtained an integral equation (3.9) whose kernel vanished as $Ka \to \infty$, and which could be solved, rigorously, by iteration.

Sayer (1980) also took $N = 0$ but considered an arbitrary cylinder (subject to ∂D having properties J); he proved the following result. Let $a_0^1 = b_1 + ib_2$, where b_1 and b_2 are real. Then, the integral equation (3.9), corresponding to this choice for $G_1(P, Q)$, always has a unique solution provided that: (i) b_1 and b_2 do *not* satisfy

$$b_2 + 2\pi(b_1^2 + b_2^2) = 0, \tag{3.10}$$

and (ii) the eigenfunction of the associated interior wave–Dirichlet problem does not vanish at the origin. (Actually, Sayer considered the water to be of constant finite depth h, whence the factor 2π in (3.10) is replaced by a function of h. He also performed many numerical computations for heaving circular and elliptic cylinders, and always found (ii) to hold.)

Ursell (1981) also considered a cylinder of arbitrary cross-section. He proved the following theorem.

Theorem 3. Let ∂D have properties J and let the constants a_m^σ in (3.8) be chosen such that $\mathscr{I}(a_m^\sigma) > 0$ for $\sigma = 1, 2$ and $m = 0, 1, \ldots, N$. Then the integral equation corresponding to this choice for $G_1(P, Q)$, namely (3.9), is uniquely solvable at any given value of K, provided that N is sufficiently large.

Returning to our problem, we see that if $\phi(q)$ satisfies the null-field equations, then, by taking a suitable linear combination of these equations, $\phi(q)$ also satisfies an integral equation of the second kind, (3.9). Moreover, by theorem 3, this equation has a unique solution at any given value of K. Let us now prove the converse.

Suppose that $\phi(q)$ is the unique solution of (3.9). Then, we can use (3.7) to define a function $U_1(P_-)$ which, by (3.9), vanishes on ∂D. We cannot immediately assert that U_1 vanishes everywhere in D_-, since G_1 is singular at O. However, if $P_- \in D_N$, we can use (3.1) and rewrite (3.7) as

$$U_1(P_-) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^\sigma(P_-)\, A_m^\sigma + \sum_{m=0}^{N} \sum_{\sigma=1}^{2} a_m^\sigma\, \Phi_m^\sigma(P_-)\, A_m^\sigma,$$

where

$$A_m^\sigma = \int_{\partial D} \left\{ \Phi_m^\sigma(q)\, V(q) - \phi(q) \frac{\partial}{\partial n_q} \Phi_m^\sigma(q) \right\} ds_q.$$

Now, we wish to show that $\phi(q)$ satisfies (N.F.), i.e. that $A_m^\sigma = 0$ for $\sigma = 1, 2$ and $m = 0, 1, 2, \ldots$. This can be proved by using an argument given by Ursell (1981). Consider the integral

$$I \equiv \int_{C_-} \left\{ U_1 \frac{\partial}{\partial n} U_1^* - U_1^* \frac{\partial}{\partial n} U_1 \right\} ds, \tag{3.11}$$

where the asterisk denotes the complex conjugate. Since U_1 and U_1^* vanish on ∂D, an application of Green's theorem in $D_- \backslash D_N$ shows that $I = 0$. Ursell (1981) then proved that if $\mathscr{I}(a_m^\sigma) > 0$, then I can only vanish if

$$A_m^\sigma = 0 \quad \text{for} \quad \sigma = 1, 2 \quad \text{and} \quad m = 0, 1, 2, \ldots, N.$$

If we let $N \to \infty$, we see that $U_1(P_-)$ vanishes everywhere in D_- and that $\phi(q)$ satisfies the null-field equations.† We have thus proved the following theorem.

Theorem 4. Suppose that ∂D has properties J and that $V(q)$ is continuous on ∂D. Then, the null-field equations for water waves in two dimensions, (N.F.), possess a unique solution for all values of K.

Corollary. If $\phi(q)$ satisfies the null-field equations (or the integral equation (3.9)), the solution of $\mathscr{P}$ is given by

$$2\pi\phi(P) = \int_{\partial D} \left\{ G_1(P, q)\, V(q) - \phi(q) \frac{\partial}{\partial n_q} G_1(P, q) \right\} ds_q. \tag{3.12}$$

Proof. We simply replace U_0 and G_0 by U_1 and G_1, respectively, in the proof of theorem 2. In that proof, the restriction to regular values of K was needed to ensure

† The proof that $U_1(P_-) \equiv 0$ given by Martin (1980) for the exterior problems of acoustics is incorrect; however, similar arguments to those used above can be used to construct a valid proof, and all the results of that paper remain unaltered.

that $U_0(P_-) \equiv 0$. Here, we have already shown that $U_1(P_-) \equiv 0$ for any value of K, and so (3.12) holds for all values of K.

We remark that, when P is exterior to C_+, the null-field equations imply that the representation (3.12) reduces to (3.3) (with (3.4)).

4. A half-immersed circular cylinder; the method of multipoles

Consider a half-immersed circular cylinder, with wetted surface C, floating in the free surface of deep water. We define circular polar co-ordinates (r, θ) such that points on C have co-ordinates (a, θ) with $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. The symmetric boundary-value problem $\mathscr{P}$ corresponding to heaving oscillations of the cylinder was first solved by Ursell (1949). We shall now briefly describe his method. For this particular geometry, we can represent $\phi(P)$ for all $P \in D$ as an infinite series of multipole potentials. Thus, we write

$$\phi(r,\theta) = \sum_{n=0}^{\infty} c_n \Phi_n^1(r,\theta) \quad (r \geqslant a, \ -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi), \tag{4.1}$$

where Φ_0^1 is the potential due to a simple source at the origin and Φ_n^1 are symmetric wave-free potentials for $n > 0$; see appendix A. Equation (4.1) satisfies Laplace's equation in D, the free-surface condition (1.1) and the radiation condition. Equation (4.1) also satisfies the boundary condition (1.2) on C if the coefficients c_n can be chosen such that

$$V(a,\theta) = U_0 \cos\theta = \sum_{n=0}^{\infty} c_n \left\langle \frac{\partial}{\partial r} \Phi_n^1(r,\theta) \right\rangle \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi), \tag{4.2}$$

where U_0 is a constant, and angular brackets indicate that r is to be put equal to a.

To find the unknown coefficients c_n, multiply (4.2) by the complete set $\{\cos 2m\theta\}$, $m = 0, 1, \ldots$, and integrate over C to give

$$\int_0^{\frac{1}{2}\pi} V(a,\theta)\cos 2m\theta \, a\, d\theta = \sum_{n=0}^{\infty} c_n \int_0^{\frac{1}{2}\pi} \left\langle a \frac{\partial}{\partial r} \Phi_n^1(r,\theta) \right\rangle \cos 2m\theta \, d\theta \quad (m = 0, 1, 2, \ldots).$$

This leads to an infinite system of linear algebraic equations for c_n; approximate values for c_n may be obtained by numerically solving a truncated system of equations.

Instead of multiplying (4.2) by $\{\cos 2m\theta\}$, let us multiply by the complete set $\{\Phi_m^1(a,\theta)\}$, $m = 0, 1, \ldots$, and integrate over C. We obtain

$$\int_0^{\frac{1}{2}\pi} V(a,\theta)\, \Phi_m^1(a,\theta)\, a\, d\theta = \sum_{n=0}^{\infty} c_n \int_0^{\frac{1}{2}\pi} \left\langle a \frac{\partial}{\partial r} \Phi_n^1(r,\theta) \right\rangle \Phi_m^1(a,\theta)\, d\theta. \tag{4.3}$$

We now apply Green's theorem to Φ_n^1 and Φ_m^1 in the region bounded by C, F and S, where S is a large semicircle of radius r_∞. There is no contribution from the integration over F, and the contribution from the integration over S also vanishes as $r_\infty \to \infty$, by (3.5). Hence (4.3) becomes

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} V(a,\theta)\, \Phi_m^1(a,\theta)\, a\, d\theta &= \sum_{n=0}^{\infty} c_n \int_0^{\frac{1}{2}\pi} \left\langle a \frac{\partial}{\partial r} \Phi_m^1(r,\theta) \right\rangle \Phi_n^1(a,\theta)\, d\theta \\ &= \int_0^{\frac{1}{2}\pi} \phi(a,\theta) \left\langle a \frac{\partial}{\partial r} \Phi_m^1(r,\theta) \right\rangle d\theta \quad (m = 0, 1, \ldots),\end{aligned} \tag{4.4}$$

by (4.1). We see that (4.4) are precisely the null-field equations for the symmetric oscillations of a half-immersed circular cylinder. This may be compared with the corresponding exterior problem of acoustics. For an oscillating circular cylinder, the null-field equations simply yield the Fourier components of the well known exact solution; for all other geometries, the null-field equations of acoustics must be solved numerically. For water-wave problems, the null-field equations must always be solved numerically; this will be discussed in § 5.

5. Numerical solution of the null-field equations

The null-field equations may be written as

$$\int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} \Phi_m^\sigma(q)\, ds_q = V_m^\sigma \quad (\sigma = 1, 2;\; m = 0, 1, \ldots), \tag{5.1}$$

where $\phi(q)$ is to be determined and the known constants V_m^σ are given by

$$V_m^\sigma = \int_{\partial D} V(q)\, \Phi_m^\sigma(q)\, ds_q.$$

In § 4, we showed that when ∂D is a semicircle, the null-field equations are related to Ursell's method of multipoles; even for this simplest of geometries, the equations must be solved numerically.

One approach is to write

$$\phi(q) = \sum_{n=0}^{\infty} a_n \phi_n(q), \tag{5.2}$$

where the set of functions $\{\phi_n(q)\}$ is required to be complete over ∂D. Substituting (5.2) into (5.1), we obtain an infinite system of linear algebraic equations for the unknown coefficients a_n; truncating this system leads to a numerical method for solving the null-field equations. The choice of the set $\{\phi_n\}$ is largely at our disposal; trigonometric functions and Chebyshev polynomials are suitable, but many other choices could be made. Rather than give a discussion of general numerical aspects here, we shall describe a particular example.

We consider the vertical oscillations of a half-immersed elliptic cylinder. The corresponding boundary-value problem $\mathscr{P}$ has been treated by several authors. For example, Porter (1960) has used conformal mapping and the method of multipoles, whilst Kim (1965) has solved the source integral equation (2.8).

Let an arbitrary point $q \equiv (x, y)$ on the wetted surface of the cylinder have co-ordinates

$$x = a \sin\eta, \quad y = b\cos\eta \quad (-\tfrac{1}{2}\pi \leqslant \eta \leqslant \tfrac{1}{2}\pi),$$

where $2a$ and b are the beam and draught, respectively, of the cylinder. In circular polar co-ordinates (r, θ), we have

$$x = r\sin\theta, \quad y = r\cos\theta \quad (-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi),$$

where

$$r = b(\cos^2\eta + H^2\sin^2\eta)^{\frac{1}{2}}, \quad \tan\theta = H\tan\eta, \quad H = a/b.$$

Since the motion is symmetric about $\theta = 0\,(\eta = 0)$, we only require the even $(\sigma = 1)$ null-field equations, and only need to integrate over half of ∂D. Moreover, we have $V(q)\, ds_q = U_0 a \cos\eta\, d\eta$, where the cylinder is oscillating with vertical velocity

$$\mathscr{R}\{U_0 e^{-i\omega t}\}.$$

Thus the null-field equations become

$$\int_0^{\frac{1}{2}\pi} \phi(\eta)\left\{b\frac{\partial}{\partial n}\Phi_m^1(q)\right\}(\sin^2\eta + H^2\cos^2\eta)^{\frac{1}{2}}\,d\eta = U_0 a V_m \quad (m = 0, 1, 2, \ldots), \tag{5.3}$$

where

$$V_m = \int_0^{\frac{1}{2}\pi} \Phi_m^1(r,\theta)\cos\eta\,d\eta.$$

The source potential Φ_0^1 may be evaluated using the expansion given by Yu & Ursell (1961), namely

$$\Phi_0^1(r,\theta) = -\{(\log Kr - i\pi + \gamma)\cos(Kr\sin\theta) + \theta\sin(Kr\sin\theta)\}\exp(-Kr\cos\theta) + \sum_{m=1}^{\infty}\frac{(-Kr)^m}{m!}\left\{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{m}\right\}\cos m\theta,$$

where $\gamma = 0{\cdot}5772\ldots$ is Euler's constant. The wave-free potentials Φ_m^1 are given by (A 1) as

$$\Phi_m^1(r,\theta) = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}} \quad (m = 1, 2, \ldots).$$

The normal derivative of Φ_m^1, $m = 0, 1, 2, \ldots$, may be evaluated using

$$\frac{\partial}{\partial n}\Phi(r,\theta) = \cos(\alpha-\theta)\frac{\partial\Phi}{\partial r} + \sin(\alpha-\theta)\frac{1}{r}\frac{\partial\Phi}{\partial\theta},$$

where $H\tan\alpha = \tan\eta$.

To solve the null-field equations, we substitute

$$\phi(\eta) = U_0 a \sum_{n=0}^{N} a_n\phi_n(\eta)$$

into the first $N+1$ equations of (5.3) to obtain

$$\sum_{n=0}^{N} K_{mn}a_n = V_m \quad (m = 0, 1, \ldots, N), \tag{5.4}$$

where

$$K_{mn} = \int_0^{\frac{1}{2}\pi}\phi_n(\eta)\left\{b\frac{\partial}{\partial n}\Phi_m^1(r,\theta)\right\}(\sin^2\eta + H^2\cos^2\eta)^{\frac{1}{2}}\,d\eta.$$

(5.4) is a system of $N+1$ linear algebraic equations for $N+1$ unknown coefficients a_n; V_m and K_{mn} may be evaluated numerically using any suitable quadrature formula (the integrands are non-singular). In our numerical work, we tried $\phi_n(\eta) = \cos 2n\eta$ and $\phi_n(\eta) = T_{2n}(2\eta/\pi)$; although other choices could have been made, we found the Chebyshev polynomials to be quite satisfactory for our problem. (Note that in numerical computations the restriction to complete sets of functions $\{\phi_n\}$ is probably not required.)

In table 1, we give values of the virtual-mass coefficient for various values of Ka and H, where

$$\text{virtual-mass coefficient} = \frac{-4}{\pi}\int_0^{\frac{1}{2}\pi}\mathscr{R}\left\{\frac{\phi(\eta)}{U_0 a}\right\}\cos\eta\,d\eta.$$

(Here the virtual mass has been normalized by the mass of the fluid displaced by a half-immersed circular cylinder of radius a.) All the results shown were obtained using

H	Wavenumber Ka 0·5	1·0	1·5	2·0	2·5	3·0
0·5	0·5369	0·6805	0·7868	0·8470	0·8828	0·9056
0·75	0·5782	0·6142	0·7056	0·7750	0·8228	0·8561
1·0	0·6446	0·6050	0·6649	0·7266	0·7759	0·8135
1·25	0·7061	0·6196	0·6512	0·6997	0·7444	0·7815
1·5	0·7578	0·6423	0·6522	0·6878	0·7257	0·7598
1·75	0·8005	0·6663	0·6603	0·6849	0·7161	0·7462
2·0	0·8360	0·6891	0·6715	0·6873	0·7124	0·7385
2·25	0·8656	0·7100	0·6837	0·6926	0·7124	0·7349
2·5	0·8907	0·7287	0·6960	0·6993	0·7147	0·7340
2·75	0·9122	0·7454	0·7077	0·7065	0·7184	0·7348
3·0	0·9307	0·7603	0·7186	0·7139	0·7228	0·7368

TABLE 1. Virtual-mass coefficient for heaving elliptic cylinder, for various Ka and H, where $H = a/b$ = half-beam/draught

Chebyshev polynomials and $N \leqslant 7$. Comparing our numerical values with the graphical results of Porter (1960, figure 13) and Kim (1965, figure 15), we see that the agreement is good (Kim's result must be multiplied by $2/\pi$). We remark, however, that although the null-field equations are guaranteed to have a unique solution, our simple numerical scheme for solving these equations does not converge for very thin ellipses ($H < 0{\cdot}4$ and $H > 3$, approximately). This difficulty also occurs when solving the null-field equations of acoustics, but can be alleviated by using different numerical methods; for references, see e.g. Martin (1980).

6. Three dimensions

We shall conclude this paper by briefly describing how the results of § 3 may be extended to water-wave problems in three dimensions. To do this, we shall require the expansion of the simple wave source $G_0(P, Q)$ (given by (2.1*b*)), which is analogous to (3.1). For $r_Q > r_P$, we have (this is proved in appendix B)

$$G_0(P,Q) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_{jm}^{\sigma}(P)\, \Phi_{jm}^{\sigma}(Q), \tag{6.1}$$

where the functions α_{jm}^{σ} and Φ_{jm}^{σ} are defined in appendix B; as before, α_{jm}^{σ} are regular potentials which satisfy the free-surface condition (1.1), whilst Φ_{jm}^{σ} are multipole potentials which satisfy the free-surface and radiation conditions, and are singular at O.

We are now in a position to state the null-field equations for water waves in three dimensions. Consider a rigid body floating in the free surface of deep water. Then, following the derivation given in § 3, we easily obtain

$$\int_{\partial D} \left\{ \phi(q) \frac{\partial}{\partial n_q} \Phi_{jm}^{\sigma}(q) - V(q)\, \Phi_{jm}^{\sigma}(q) \right\} ds_q = 0 \quad (\sigma = 1, 2;\ m, j = 0, 1, 2, \ldots). \tag{6.2}$$

(We have assumed that the wetted surface of the floating body, ∂D, is bounded and has properties J.)

The existence of a unique solution to this system of equations is guaranteed by the following theorem.

Theorem 5. Suppose that ∂D is a bounded surface with properties J and that $V(q)$ is continuous on ∂D. Then, the null-field equations for water waves in three dimensions, (6.2), possess a unique solution for all values of K.

Proof. We proceed exactly as in the two-dimensional case, i.e. we show that $\phi(q)$ satisfies (6.2) if and only if $\phi(q)$ satisfies a Fredholm integral equation of the second kind which is known to possess a unique solution. In order to obtain such an equation, we extend the results of Ursell (1981) to three dimensions. This is straightforward, once the expansion (6.1) has been found, e.g. integrals of the form (3.11) are easily evaluated, see appendix B. Thus, we are able to prove a three-dimensional analogue of theorem 3. All the remaining arguments used to prove theorem 4 may now be used here to complete the proof of theorem 5.

Let us now make a few remarks on the solution of the null-field equations in three dimensions. If ∂D is a hemisphere, the null-field equations are equivalent to the method of multipoles, as used by Havelock (1955) for the heaving half-immersed sphere. If ∂D is axisymmetric (about the y-axis), it should be possible to evaluate analytically some of the integrals appearing in (6.2); this would reduce the surface integrals over ∂D to line integrals. Similarly, any other symmetries in a particular problem $\mathscr{P}$ should also be exploited. Finally, most of the comments made in § 5 are also pertinent here.

7. Conclusions

The best-known method for treating water-wave radiation problems (in two and three dimensions) is to solve an integral equation of the second kind over the (mean) wetted surface. However, it is also well known (John 1950) that the usual boundary integral equations are not uniquely solvable at the irregular values of K. In this paper we have described an alternative method, which is to solve the infinite system of null-field equations. These equations appear to be new in the context of water-wave problems, although there is an extensive literature on the corresponding equations for radiation and scattering problems in acoustics, electromagnetism and elastodynamics; see Waterman (1969) and Martin (1980) for references.

We have shown that the null-field equations for water waves always have a unique solution – the unphysical irregular values do not occur. Moreover, this solution may be used to solve the original boundary-value problem $\mathscr{P}$. We have proved these results in two and three dimensions for water of infinite depth; the extension to water of constant finite depth is easily made, using the corresponding multipole potentials, $\tilde{\Phi}^{\sigma}_{m}$ and $\tilde{\Phi}^{\sigma}_{jm}$, defined in appendices A and B, respectively.

In § 5, we described a simple exact method for reducing the null-field equations to an infinite system of linear algebraic equations. This method may be used to solve the null-field equations, numerically, by making two approximations: the infinite set of equations must first be truncated and then the unknown potential must be approximated by a finite combination of the chosen basis functions. As an example, we used this method to solve the two-dimensional problem $\mathscr{P}$ corresponding to the vertical oscillations of a half-immersed elliptic cylinder.

It is clear that much work remains to be done on the numerical solution of the null-field equations. Nevertheless, the null-field method has already proved to be computationally useful in other branches of mathematical physics; it is therefore hoped that the null-field equations for water waves will be just as useful.

I should like to thank Professor Fritz Ursell for many interesting discussions, and the Science Research Council for their financial support.

Appendix A. Expansion of the source potential in two dimensions

In this appendix, we shall simply quote a theorem which has been proved by Ursell (1981).

Theorem A. When $r_P < r_Q$, the source potential defined by (2.1 a) can be expanded as

$$G_0(P,Q) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^\sigma(P)\, \Phi_m^\sigma(Q),$$

where

$$\Phi_0^1(P) = \int_0^\infty e^{-ky} \cos kx \frac{dk}{k-K}, \quad \Phi_0^2(P) = -\frac{\partial}{\partial x}\Phi_0^1(P),$$

$$\Phi_m^1(P) = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}, \tag{A 1}$$

$$\Phi_m^2(P) = \frac{\sin(2m+1)\theta}{r^{2m+1}} + \frac{K}{2m}\frac{\sin 2m\theta}{r^{2m}},$$

$$\alpha_0^1(P) = -2e^{-Ky}\cos Kx, \quad \alpha_0^2(P) = \frac{-2}{K} e^{-Ky} \sin Kx,$$

$$\alpha_m^1(P) = \frac{-2(2m-1)!}{K^{2m}} \sum_{q=2m}^{\infty} \frac{(-Kr)^q}{q!} \cos q\theta,$$

$$\alpha_m^2(P) = \frac{2(2m)!}{K^{2m+1}} \sum_{q=2m+1}^{\infty} \frac{(-Kr)^q}{q!} \sin q\theta,$$

$m = 1, 2, \ldots$, and the point $P \equiv (x, y)$ has circular polar co-ordinates given by

$$x = r\sin\theta, \quad y = r\cos\theta$$

(with $r = r_P$).

We remark that theorem A may also be proved for a simple wave source in water of constant finite depth; for this case, the α_m^σ remain unaltered whilst the infinite-depth multipole potentials Φ_m^σ must be replaced by the corresponding potentials for finite depth, $\tilde{\Phi}_m^\sigma$; see Ursell (1981).

Appendix B. Expansion of the source potential in three dimensions

The simple wave source in three dimensions, $G_0(P,Q)$ is defined by (2.1b); the properties of $G_0(P,Q)$ are given by Wehausen & Laitone (1960, p. 475). We introduce spherical polar co-ordinates for $P \equiv (x,y,z)$ and $Q \equiv (\xi, \eta, \zeta)$, and write

$$x = \rho\cos\alpha, \quad y = r\cos\theta, \quad z = \rho\sin\alpha, \quad \rho = r\sin\theta,$$

$$\xi = \rho_0\cos\alpha_0, \quad \eta = r_0\cos\theta_0, \quad \zeta = \rho_0\sin\alpha_0, \quad \rho_0 = r_0\sin\theta_0.$$

(For convenience, we have used r and r_0 instead of r_P and r_Q, respectively.)

For large $R = ((x-\xi)^2 + (z-\zeta)^2)^{\frac{1}{2}}$, we have

$$\begin{aligned} G_0(P,Q) &= \pi i K e^{-K(y+\eta)} H_0^{(1)}(KR) + O(R^{-1}) \\ &= \pi i K e^{-K(y+\eta)} \sum_{m=0}^{\infty} \epsilon_m H_m^{(1)}(K\rho_0) J_m(K\rho) \cos m(\alpha-\alpha_0) + O(R^{-1}), \end{aligned} \tag{B 1}$$

where we have supposed that $\rho < \rho_0$ and then used the well known expansion for $H_0^{(1)}(KR)$. (ϵ_m is the Neumann factor, defined by $\epsilon_0 = 1$, $\epsilon_m = 2$ for $m > 0$.) The expansion of $G_0(P, Q)$ that follows has a form that is suggested by the behaviour of G_0 at ∞, i.e. by (B 1).

From (2.1*b*), the potential of a simple wave source at O is

$$G_0(P, 0) = \frac{1}{r} + K \int_0^\infty e^{-ky} J_0(k\rho) \frac{dk}{k-K} = \Psi_0^0(\rho, y),$$

say. Potentials which have a singularity of a higher order at O have been considered by Thorne (1953). He considers functions of the form $\phi_n^m(\rho, y)\, E_m^\sigma(\alpha)$, where

$$\phi_n^m(\rho, y) = \frac{P_n^m(\cos\theta)}{2r^{n+1}}\{1 + (-1)^{m+n}\} + \frac{K(-1)^{m+n}}{(n-m)!} \int_0^\infty k^n e^{-ky} J_m(k\rho) \frac{dk}{k-K}, \tag{B 2}$$

$$E_m^1(\alpha) = \epsilon_m^{\frac{1}{2}} \cos m\alpha, \quad E_m^2(\alpha) = 2^{\frac{1}{2}} \sin m\alpha.$$

($P_n^m(x)$ are the associated Legendre functions, defined by

$$P_n^m(x) = (1-x^2)^{\frac{1}{2}m} \frac{d^m}{dx^m} P_n(x) \quad (m \leqslant n);$$

this definition is used by Thorne (1953), but differs by a factor of $(-1)^m$ from that used by Erdélyi *et al.* (1953).) Let us write

$$\Psi_0^m(\rho, y) \equiv \phi_m^m(\rho, y) = \int_0^\infty k^{m+1} e^{-ky} J_m(k\rho) \frac{dk}{k-K},$$

where we have used (Erdélyi *et al.* 1953, § 7.8, equation (10))

$$\frac{P_n^m(\cos\theta)}{r^{n+1}} = \frac{1}{(n-m)!} \int_0^\infty k^n e^{-ky} J_m(k\rho)\, dk. \tag{B 3}$$

It follows that, for all $m \geqslant 0$,

$$\Psi_0^m(\rho, y) = \pi i K^{m+1} H_m^{(1)}(K\rho)\, e^{-Ky} + O(\rho^{-1}), \tag{B 4}$$

as $\rho \to \infty$. Comparing (B 4) with (B 1), we see that

$$G_0(P, Q) = \sum_{m=0}^\infty \sum_{\sigma=1}^2 \alpha_{0m}^\sigma(P)\, \Phi_{0m}^\sigma(Q) + R_0(P, Q), \tag{B 5}$$

where

$$\Phi_{0m}^\sigma(P) = \Psi_0^m(\rho, y)\, E_m^\sigma(\alpha), \tag{B 6}$$

$$\alpha_{0m}^\sigma(P) = K^{-m} e^{-Ky} J_m(K\rho)\, E_m^\sigma(\alpha), \tag{B 7}$$

and $R_0(P, Q) = O(R^{-1})$ as $R \to \infty$.

$R_0(P, Q)$, the 'remainder' in (B 5), is wave-free, and so we shall seek an expansion of R_0 in terms of 'wave-free potentials'; these have been constructed by Havelock (1955) and are given by

$$\Phi_{jm}^\sigma(P) = \Psi_j^m(\rho, y)\, E_m^\sigma(\alpha), \tag{B 8}$$

where

$$\Psi_j^m \equiv \phi_{m+2j}^m + \frac{K}{2j}\phi_{m+2j-1}^m = \frac{P_{m+2j}^m(\cos\theta)}{r^{m+2j+1}} + \frac{K}{2j}\frac{P_{m+2j-1}^m(\cos\theta)}{r^{m+2j}}$$

$$= \frac{1}{(2j)!}\int_0^\infty k^{m+2j-1}(k+K)\,e^{-ky}\,J_m(k\rho)\,dk \quad (j \geqslant 1). \quad \text{(B 9)}$$

We can now state our theorem on the expansion of $G_0(P, Q)$.

Theorem B. When $r_P < r_Q$ (i.e. $r < r_0$), the source potential defined by (2.1*b*) can be expanded as

$$G_0(P,Q) = \sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\sigma=1}^{2} \alpha_{jm}^{\sigma}(P)\,\Phi_{jm}^{\sigma}(Q), \quad \text{(B 10)}$$

where

$$\alpha_{jm}^{\sigma}(P) = \frac{(2j)!}{K^{2j}}E_m^{\sigma}(\alpha)\,r^m \sum_{q=2j}^{\infty}\frac{(-Kr)^q}{(2m+q)!}P_{m+q}^m(\cos\theta), \quad \text{(B 11)}$$

for $\sigma = 1, 2$ and $m, j = 0, 1, 2, \ldots$.

Proof. It will be convenient to assume initially that $y < \eta$. Using (B 3) (with $n = m = 0$) in (2.1*b*), we obtain

$$G_0(P,Q) = \int_0^\infty e^{-k\eta}\,J_0(kR)\left\{\cosh ky + \frac{K e^{-ky}}{k-K}\right\}dk.$$

Expanding $J_0(kR)$ and using (B 5), we find that

$$R_0(P,Q) = \sum_{m=0}^{\infty}\epsilon_m \cos m(\alpha-\alpha_0)\int_0^\infty e^{-k\eta}\,J_m(k\rho_0)\,G_m(\rho,y;k)\,dk, \quad \text{(B 12)}$$

where

$$G_m = J_m(k\rho)\cosh ky + \frac{1}{k-K}\left\{K e^{-ky}J_m(k\rho) - \frac{k^{m+1}}{K^m}e^{-Ky}J_m(K\rho)\right\}$$

$$= \sum_{q=0}^{\infty}\frac{(1+(-1)^q)\,(kr)^{m+q}}{2(2m+q)!}P_{m+q}^m(\cos\theta)$$

$$+ \sum_{q=0}^{\infty}\frac{(-r)^q\,(kr)^m}{(2m+q)!}\frac{(Kk^q - kK^q)}{k-K}P_{m+q}^m(\cos\theta), \quad \text{(B 13)}$$

and we have used (Thorne 1953, equation (7))

$$e^{\pm ky}J_m(k\rho) = \sum_{q=0}^{\infty}\frac{(\pm 1)^q\,(kr)^{m+q}}{(2m+q)!}P_{m+q}^m(\cos\theta).$$

It is easy to see that, in (B 13), the $q = 0$ terms cancel, whilst the $q = 1$ terms vanish. For $q > 1$, we have

$$Kk^q - kK^q = (k-K)\sum_{s=1}^{q-1}k^s K^{q-s},$$

whence the second sum in (B 13) becomes

$$\sum_{s=1}^{\infty}k^{m+s}\sum_{q=s+1}^{\infty}\frac{(-1)^q\,r^{m+q}\,K^{q-s}}{(2m+q)!}P_{m+q}^m(\cos\theta),$$

where we have changed the order of summation. Write $q = 2j$ in the first sum of (B 13) and collect together terms to give

$$G_m = \sum_{j=1}^{\infty}(k+K)\,k^{m+2j-1}\sum_{q=2j}^{\infty}\frac{(-1)^q\,r^{m+q}\,K^{q-2j}}{(2m+q)!}P_{m+q}^m(\cos\theta). \quad \text{(B 14)}$$

Substituting (B 14) into (B 12), and making use of (B 9), we obtain the desired result, when $y < \eta$. However, since both sides of (B 10) are regular solutions of Laplace's equation when $r < r_0$, it follows that (B 10) holds for all P and Q such that $r_P < r_Q$. This concludes the proof of theorem B.

Theorem B may also be proved for a simple wave source in water of constant finite depth h. As in the two-dimensional case, we simply replace the infinite-depth multipole potentials Φ^{σ}_{jm} by the corresponding potentials for finite depth. It can be shown (Thorne 1953; Wang 1966) that these are given by

$$\tilde{\Phi}^{\sigma}_{jm}(P) = \tilde{\Psi}^{m}_{j}(\rho, y)\, E^{\sigma}_{m}(\alpha), \tag{B 15}$$

$$\tilde{\Psi}^{m}_{0}(\rho, y) = \int_0^\infty \frac{k^{m+1}\cosh k(h-y)}{k\sinh kh - K\cosh kh} J_m(k\rho)\, dk,$$

$$\tilde{\Psi}^{m}_{j}(\rho, y) = \Psi^{m}_{j}(\rho, y) - \frac{1}{(2j)!}\int_0^\infty \frac{e^{-kh}(K+k)\,(K\sinh ky - k\cosh ky)\, k^{m+2j-1}}{k\sinh kh - K\cosh kh} J_m(k\rho)\, dk,$$

$j = 1, 2, \ldots$, and the path of integration passes beneath the pole at $k = k_0$, where k_0 is the unique positive real root of

$$K\cosh k_0 h - k_0 \sinh k_0 h = 0.$$

We have

$$\tilde{\Psi}^{m}_{0}(\rho, y) \sim 2\pi i k_0^{m+1} A(k_0 h)\cosh k_0 h \cosh k_0(h-y)\, H^{(1)}_m(k_0\rho), \tag{B 16}$$

$$\tilde{\Psi}^{m}_{j}(\rho, y) \sim \frac{2\pi i A(k_0 h)}{(2j)!\cosh k_0 h} k_0^{m+2j+1}\cosh k_0(h-y)\, H^{(1)}_m(k_0\rho),$$

as $\rho \to \infty$, for $j = 1, 2, \ldots$, where

$$A(k_0 h) = (2k_0 h + \sinh 2k_0 h)^{-1}.$$

The integral

$$I \equiv \int_{C_-}\left\{\tilde{\Phi}^{\sigma}_{jm}\frac{\partial}{\partial n}\tilde{\Phi}^{\nu *}_{lk} - \tilde{\Phi}^{\nu *}_{lk}\frac{\partial}{\partial n}\tilde{\Phi}^{\sigma}_{jm}\right\} ds$$

is typical of those required in § 6. To evaluate it, we apply Green's theorem to $\tilde{\Phi}^{\sigma}_{jm}$ and $\tilde{\Phi}^{\nu *}_{lk}$, in the region bounded by C_-, F, the bottom $y = h$, and the surface of a large vertical cylinder of radius ρ, whose axis coincides with the y-axis; we find that

$$\begin{aligned} I &= -\int_0^h \left\{\tilde{\Psi}^{m}_{j}(\rho, y)\frac{\partial}{\partial n}\tilde{\Psi}^{k*}_{l} - \tilde{\Psi}^{k*}_{l}\frac{\partial}{\partial n}\tilde{\Psi}^{m}_{j}\right\}\rho\, dy \int_0^{2\pi} E^{\sigma}_{m}(\alpha)\, E^{\nu}_{k}(\alpha)\, d\alpha \\ &= 2\pi\delta_{mk}\delta_{\sigma\nu}\int_0^h \left\{\tilde{\Psi}^{m}_{j}(\rho, y)\frac{\partial}{\partial \rho}\tilde{\Psi}^{m*}_{l} - \tilde{\Psi}^{m*}_{l}\frac{\partial}{\partial \rho}\tilde{\Psi}^{m}_{j}\right\}\rho\, dy, \end{aligned}$$

where we have used (B 15) and integrated over α. As an example, let us take $j = l = 0$; if we use (B 16), evaluate the elementary integral over y, and use the Wronskian (Erdélyi *et al.* 1953, § 7.11, equation (30))

$$H^{(1)}_m(z)\, H^{(2)\prime}_m(z) - H^{(2)}_m(z)\, H^{(1)\prime}_m(z) = -4i/\pi z,$$

we find that

$$\begin{aligned} I &= -8\pi^2 i A(k_0 h)\, k_0^{2m+1}\cosh^2 k_0 h\, \delta_{mk}\delta_{\sigma\nu} \\ &\sim -8\pi^2 i\, K^{2m+1}\delta_{mk}\delta_{\sigma\nu} \quad \text{as} \quad h \to \infty. \end{aligned}$$

Other combinations of j and l lead to similar integrals.

REFERENCES

Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. 1953 *Higher Transcendental Functions*, vol. 2. McGraw-Hill.

Havelock, T. H. 1955 Waves due to a floating sphere making periodic heaving oscillations. *Proc. R. Soc. Lond.* A **231**, 1–7.

John, F. 1950 On the motion of floating bodies II. *Comm. Pure Appl. Math.* **3**, 45–101.

Kim, W. D. 1965 On the harmonic oscillations of a rigid body on a free surface. *J. Fluid Mech.* **21**, 427–451.

Kleinman, R. E. & Roach, G. F. 1974 Boundary integral equations for the three-dimensional Helmholtz equation. *SIAM Rev.* **16**, 214–236.

Martin, P. A. 1980 On the null-field equations for the exterior problems of acoustics. *Quart. J. Mech. Appl. Math.* **33**, 385–396.

Mei, C. C. 1978 Numerical methods in water-wave diffraction and radiation. *Ann. Rev. Fluid Mech.* **10**, 393–416.

Porter, W. R. 1960 Pressure distributions, added-mass, and damping coefficients for cylinders oscillating in a free surface. Dissertation, University of California, Berkeley.

Sayer, P. 1980 An integral-equation method for determining the fluid motion due to a cylinder heaving on water of finite depth. *Proc. R. Soc. Lond.* A **372**, 93–110.

Thorne, R. C. 1953 Multipole expansions in the theory of surface waves. *Proc. Camb. Phil. Soc.* **49**, 707–716.

Ursell, F. 1949 On the heaving motion of a circular cylinder on the surface of a fluid. *Quart. J. Mech. Appl. Math.* **2**, 218–231.

Ursell, F. 1950 Surface waves on deep water in the presence of a submerged circular cylinder. II. *Proc. Camb. Phil. Soc.* **46**, 153–158.

Ursell, F. 1953 Short surface waves due to an oscillating immersed body. *Proc. R. Soc. Lond.* A **220**, 90–103.

Ursell, F. 1981 Irregular frequencies and the motion of floating bodies. *J. Fluid Mech.* **105**, 143–156.

Wang, S. 1966 The hydrodynamic forces and pressure distributions for an oscillating sphere in a fluid of finite depth. Dissertation, Massachusetts Institute of Technology.

Waterman, P. C. 1969 New formulation of acoustic scattering. *J. Acoust. Soc. Am.* **45**, 1417–1429.

Wehausen, J. V. & Laitone, E. V. 1960 Surface waves. *Handbuch der Physik*, vol. 9, pp. 446–778. Springer.

Yu, Y. S. & Ursell, F. 1961 Surface waves generated by an oscillating circular cylinder on shallow water: theory and experiment. *J. Fluid Mech.* **11**, 529–551.